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Dynamical Casimir Effect for Scalar Fields I (Particle Creation)

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The dynamical Casimir effect for neutral scalar massive field in a cavity with perfect reflecting boundaries is revisited from a mathematical point of view. We consider some $1 + 1$ and $3 + 1$ dimensional examples in which the boundary oscillates. For short times, the average number of produced particles is calculated using the second order perturbation theory, and for large times, the method to calculate the number of created particles is the rotating wave approximation.

KEY WORDS: dynamical Casimir effect; rotating wave approximation; particle production.

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1. INTRODUCTION

The creation of massless particles in cavities with perfect reflecting boundaries whose walls oscillate in resonance with the eigenfrequencies of some quantum modes has been studied in some papers (Dodonov and Klimov, 1996; Crocce *et al.*, 2002; Schützhold *et al.*, 2002). In this situation the authors use several methods to calculate the production of particles (averaging over fast oscillations (Dodonov and Klimov, 1996), multiple scale analysis (Crocce *et al.*, 2002), rotating wave approximation (Law, 1994; Schützhold *et al.*, 2002). In this paper we review and discuss from a mathematical point of view the results obtained by these authors. For short times using the second order perturbation theory, and for large times, in the rotating wave approximation. Our conclusion about this topic is that, if the movement of the boundary is sufficiently smooth ($\mathcal{C}^3$ in $1 + 1$ dimensions and $\mathcal{C}^4$ in $3 + 1$ dimensions) the average number of created particles is finite, but when this movement has some type of discontinuities a divergent production of particles is possible, in agreement with the conclusions of Moore (1970) and Schützhold *et al.* (1998).

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The paper is organised as follows: In Section II we consider the neutral Klein-Gordon field in a $3 + 1$-dimensional cavity with moving boundaries, and we assume that the velocity of the wall is of order ϵ. Using an instantaneous set of eigenfunctions (see Law, 1994; Schützhold *et al.*, 1998) the Hamiltonian and the energy of the system are calculated.

Once we have quantised the field, we calculate the time-evolved vacuum state in the same way as Schützhold *et al.* (1998), and then we obtain, until order ϵ^2, the average number of produced particles.

In Section 3 some examples in $1 + 1$-dimensions are studied. For short times, the formulae obtained in Section 2 are used to calculate the average number of produced particles in an oscillating cavity. For large times, the number of produced particles in an oscillating cavity is obtained in the rotating wave approximation.

In Section 4 the dynamical Casimir effect is studied in a $3 + 1$-dimensional cavity with an oscillating boundary. Some examples are discussed, one resonant mode, two coupled non-resonant modes, etc. . . In all cases, the number of produced particles is calculated in the rotating wave approximation.

2. CANONICAL FORMULATION

2.1. Hamiltonian and Energy

We consider a neutral massive scalar field in a cavity Ω_t with perfect reflecting moving boundaries. We assume that the boundary is at rest for times smaller than 0 and returns to the initial position at time T. We also suppose that their velocity is of the order ϵ (where ϵ is a small dimensionless parameter). Then, the Hamiltonian of the system is (see Schützhold *et al.*, 1998; Schaller *et al.*, 2002)

$$\begin{aligned} H(t;\epsilon) &\equiv \sum_{\mathbf{n}} P_{\mathbf{n}}\dot{Q}_{\mathbf{n}} - L(t;\epsilon) \\ &= \frac{1}{2}\sum_{\mathbf{n}}\left(P_{\mathbf{n}}^2 + \omega_{\mathbf{n}}^2(t;\epsilon)Q_{\mathbf{n}}^2\right) + \sum_{\mathbf{n},\mathbf{k}} P_{\mathbf{n}}M_{\mathbf{nk}}(t;\epsilon)Q_{\mathbf{k}}, \end{aligned} \tag{1}$$

with

$$P_{\mathbf{n}} \equiv \frac{\partial L}{\partial \dot{Q}_{\mathbf{n}}} = \dot{Q}_{\mathbf{n}} + \sum_{\mathbf{k}} Q_{\mathbf{k}}M_{\mathbf{kn}}(t;\epsilon) \quad \text{and}$$

$$M_{\mathbf{nk}}(t;\epsilon) = \int_{\Omega_t} d^3\mathbf{x}\, \dot{f}_{\mathbf{n}}(t,\mathbf{x};\epsilon) f_{\mathbf{k}}(t,\mathbf{x};\epsilon), \tag{2}$$

where we have introduced a real complete orthonormal set of functions $f_{\mathbf{n}}(t, \mathbf{x}; \epsilon)$ satisfying the eigenvalue problem

$$\begin{cases} c^2\hbar^2\Delta_{\mathbf{x}} f_{\mathbf{n}} - m^2c^4 f_{\mathbf{n}} + \hbar^2\omega_{\mathbf{n}}^2(t;\epsilon) f_{\mathbf{n}} = 0 \\ f_{\mathbf{n}|\partial\Omega_t} = 0. \end{cases} \tag{3}$$

To calculate the energy density of the system, we need the Lagrangian density of the system

$$\mathcal{L}(t, \mathbf{x}) = \frac{1}{2}(\hbar^2(\partial_t\phi)^2 - c^2\hbar^2|\nabla_{\mathbf{x}}\phi|^2 - m^2c^4\phi^2);$$
$$\forall \mathbf{x} \in \Omega_t \subset \mathbb{R}^3 \quad \text{and} \quad \forall t \in \mathbb{R}. \tag{4}$$

If we use the canonical conjugated momentum

$$\xi(t, \mathbf{x}) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t\phi)} = \hbar^2\partial_t\phi(t, \mathbf{x}), \tag{5}$$

the energy density is given by

$$\mathcal{E}(t, \mathbf{x}) \equiv \xi\partial_t\phi - \mathcal{L}(t, \mathbf{x}) = \frac{1}{2}\left(\frac{\xi^2}{\hbar^2} + c^2\hbar^2|\nabla_{\mathbf{x}}\phi|^2 + m^2c^4\phi^2\right), \tag{6}$$

and the energy is $E(t;\epsilon) \equiv \int_{\Omega_t} d^3\mathbf{x}\mathcal{E}(t, \mathbf{x})$.

From the expansions

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{n}} \frac{Q_{\mathbf{n}}(t)}{\hbar} f_{\mathbf{n}}(t, \mathbf{x}; \epsilon) \quad \text{and} \quad \xi(t, \mathbf{x}) = \sum_{\mathbf{n}} \hbar P_{\mathbf{n}}(t) f_{\mathbf{n}}(t, \mathbf{x}; \epsilon), \tag{7}$$

we obtain

$$E(t;\epsilon) = \frac{1}{2}\sum_{\mathbf{n}} \left(P_{\mathbf{n}}^2 + \omega_{\mathbf{n}}^2(t;\epsilon) Q_{\mathbf{n}}^2\right). \tag{8}$$

The equations (1) and (8) show that the energy is not the Hamiltonian of the system.

Remark 2.1. The Hamiltonian of the system can also be obtained as follows: Firstly we transform the moving boundary into a fixed one making a **not conformal** change of coordinates

$$\mathcal{R} : (s, \mathbf{u}) \to (t(s, \mathbf{u}), \mathbf{x}(s, \mathbf{u})) = (s, \mathbf{R}(s, \mathbf{u})), \tag{9}$$

that transform the domain Ω_t into a domain $\widetilde{\Omega}$ independent of the time.

Making use of the coordinates $(s, \mathbf{u})$, the action of the system behaves

$$S = \int_{\mathbb{R}} \int_{\widetilde{\Omega}} \widetilde{\mathcal{L}}(s, \mathbf{u}) d^3\mathbf{u} ds, \tag{10}$$

with $\widetilde{\mathcal{L}}(s, \mathbf{u}) \equiv J\mathcal{L}(\mathcal{R}(s, \mathbf{u}))$, where we have introduced the Jacobian of the change J, defined by $d^3\mathbf{x} \equiv J d^3\mathbf{u}$. Let's consider the function $\widetilde{\phi}$ defined by $\widetilde{\phi}(s, \mathbf{u}) \equiv \sqrt{J}\phi(\mathcal{R}(s, \mathbf{u}))$. Then, the canonical conjugated momentum is

$$\widetilde{\xi}(s, \mathbf{u}) \equiv \frac{\partial \widetilde{\mathcal{L}}}{\partial(\partial_s \widetilde{\phi})} = \hbar^2 \left(\partial_s \widetilde{\phi} - \frac{1}{2}\widetilde{\phi}\partial_s(\ln J) + < \mathbf{u}_t, \nabla_{\mathbf{u}}\widetilde{\phi} - \frac{1}{2}\widetilde{\phi}\nabla_{\mathbf{u}}(\ln J) > \right)$$
$$= \hbar^2\sqrt{J}\partial_t\phi(\mathcal{R}(s, \mathbf{u})), \tag{11}$$

and therefore, the Hamiltonian density is

$$\widetilde{\mathcal{H}}(s, \mathbf{u}) \equiv \widetilde{\xi}\partial_s\widetilde{\phi} - \widetilde{\mathcal{L}}(s, \mathbf{u})$$
$$= \frac{1}{2}\left(\frac{\widetilde{\xi}^2}{\hbar^2} + c^2\hbar^2 J|\nabla_{\mathbf{x}}\phi|^2 + Jm^2c^4\phi^2\right) + \widetilde{\xi}(\partial_s\widetilde{\phi} - \sqrt{J}\partial_t\phi), \tag{12}$$

The Hamiltonian density in the coordinates $(t, \mathbf{x})$ is given by

$$\mathcal{H}(t, \mathbf{x}) \equiv \widetilde{\mathcal{H}}(\mathcal{R}^{-1}(t, \mathbf{x}))\frac{d^3\mathbf{u}}{d^3\mathbf{x}} = \frac{1}{J}\widetilde{\mathcal{H}}(\mathcal{R}^{-1}(t, \mathbf{x})) \tag{13}$$

Now, from the formulae (5) and (11) we have $\widetilde{\xi}(s, \mathbf{u}) = \sqrt{J}\xi(\mathcal{R}(s, \mathbf{u}))$, and an easily calculation provides

$$\mathcal{H}(t, \mathbf{x}) = \frac{1}{2}\left(\frac{\xi^2(t, \mathbf{x})}{\hbar^2} + c^2\hbar^2|\nabla_{\mathbf{x}}\phi(t, \mathbf{x})|^2 + m^2c^4\phi^2(t, \mathbf{x})\right)$$
$$+ \xi(t, \mathbf{x})\langle\partial_s\mathbf{R}(\mathcal{R}^{-1}(t, \mathbf{x})), \nabla_{\mathbf{x}}\phi(t, \mathbf{x})\rangle$$
$$+ \frac{1}{2}\xi(t, \mathbf{x})\phi(t, \mathbf{x})\partial_s(\ln J)_{|_{\mathcal{R}^{-1}(t,\mathbf{x})}}. \tag{14}$$

Finally, is not difficult to check that from this Hamiltonian density we obtain the Hamiltonian defined in formula (1).

2.2. Quantum Theory

Since we have assumed that $\Omega_t \equiv \Omega$ for $t \in (-\infty, 0] \cup [T, \infty)$, we can define, in the Schrödinger picture, the creation and annihilation operators for times smaller than zero and greater than T, in the following way (see Eq. (22) of the Schützhold *et al.*, 1998)

$$\hat{a}^\dagger_{\mathbf{n}} = \frac{1}{\sqrt{2\hbar\omega_{\mathbf{n}}(0)}}(-i\hat{P}_{\mathbf{n}} + \omega_{\mathbf{n}}(0)\hat{Q}_{\mathbf{n}}); \quad \hat{a}_{\mathbf{n}} = \frac{1}{\sqrt{2\hbar\omega_{\mathbf{n}}(0)}}(i\hat{P}_{\mathbf{n}} + \omega_{\mathbf{n}}(0)\hat{Q}_{\mathbf{n}}). \tag{15}$$

Then, the Hamiltonian operator can be written as follows:

$$\hat{H}(t;\epsilon) = \sum_{\mathbf{n}} \hbar\omega_{\mathbf{n}}(0)\left(\hat{a}_{\mathbf{n}}^{\dagger}\hat{a}_{\mathbf{n}} + \frac{1}{2}\right)$$
$$+ \frac{\hbar}{4}\sum_{\mathbf{n}} \frac{\omega_{\mathbf{n}}^2(t;\epsilon) - \omega_{\mathbf{n}}^2(0)}{\omega_{\mathbf{n}}(0)}((\hat{a}_{\mathbf{n}}^{\dagger})^2 + (\hat{a}_{\mathbf{n}})^2 + 2\hat{a}_{\mathbf{n}}^{\dagger}\hat{a}_{\mathbf{n}} + 1)$$
$$+ \frac{i\hbar}{4}\sum_{\mathbf{nk}} M_{\mathbf{nk}}(t;\epsilon)\left(\sqrt{\frac{\omega_{\mathbf{n}}(0)}{\omega_{\mathbf{k}}(0)}} - \sqrt{\frac{\omega_{\mathbf{k}}(0)}{\omega_{\mathbf{n}}(0)}}\right)(\hat{a}_{\mathbf{n}}^{\dagger}\hat{a}_{\mathbf{k}}^{\dagger} - \hat{a}_{\mathbf{n}}\hat{a}_{\mathbf{k}})$$
$$+ \frac{i\hbar}{2}\sum_{\mathbf{nk}} M_{\mathbf{nk}}(t;\epsilon)\left(\sqrt{\frac{\omega_{\mathbf{n}}(0)}{\omega_{\mathbf{k}}(0)}} + \sqrt{\frac{\omega_{\mathbf{k}}(0)}{\omega_{\mathbf{n}}(0)}}\right)\hat{a}_{\mathbf{n}}^{\dagger}\hat{a}_{\mathbf{k}}. \tag{16}$$

When the boundary moves, that is, when $t \in (0, T)$, we can define the "quasi-particle" creation and annihilation operators (see for details Grib *et al.*, 1994), by

$$\hat{\alpha}_{\mathbf{n}}^{\dagger}(t) = \frac{1}{\sqrt{2\hbar\omega_{\mathbf{n}}(t;\epsilon)}}(-i\hat{P}_{\mathbf{n}} + \omega_{\mathbf{n}}(t;\epsilon)\hat{Q}_{\mathbf{n}});$$
$$\hat{\alpha}_{\mathbf{n}}(t) = \frac{1}{\sqrt{2\hbar\omega_{\mathbf{n}}(t;\epsilon)}}(i\hat{P}_{\mathbf{n}} + \omega_{\mathbf{n}}(t;\epsilon)\hat{Q}_{\mathbf{n}}). \tag{17}$$

Then, using these operators the energy operator has the form

$$\hat{E}(t;\epsilon) = \sum_{\mathbf{n}} \hbar\omega_{\mathbf{n}}(t;\epsilon)\left(\hat{\alpha}_{\mathbf{n}}^{\dagger}(t)\hat{\alpha}_{\mathbf{n}}(t) + \frac{1}{2}\right). \tag{18}$$

Now, let $\mathcal{T}^t$ be the quantum evolution operator of the Schrödinger equation, and let $|0\rangle$ be the initial quantum state, then the average number of produced "quasi-particles" in the **n**-mode at time t is

$$\mathcal{N}_{\mathbf{n}}^m(t) \equiv \langle 0|\left(\mathcal{T}^t\right)^{\dagger}\hat{\alpha}_{\mathbf{n}}^{\dagger}(t)\hat{\alpha}_{\mathbf{n}}(t)\mathcal{T}^t|0\rangle \tag{19}$$

where m denotes the mass of the field.

If we use the identity

$$\left(\omega_{\mathbf{k}}^2(t;\epsilon) - \omega_{\mathbf{n}}^2(t;\epsilon)\right)M_{\mathbf{kn}}(t;\epsilon)$$
$$= \int_{\partial\Omega_t} < c^2(\dot{f}_{\mathbf{k}}\nabla f_{\mathbf{n}})_{|\partial\Omega_t}, \mathbf{N} > dS - 2\dot{\omega}_{\mathbf{k}}(t;\epsilon)\omega_{\mathbf{k}}(t;\epsilon)\delta_{\mathbf{nk}}, \tag{20}$$

where $\mathbf{N}$ is the outward unit normal and dS is the differential of surface, a simple calculation lead to the formula

$$\mathcal{N}_{\mathbf{n}}^{m}(t) = \sum_{\mathbf{k}} \frac{\epsilon^2 \left| \int_0^t d\tau \int_{\partial\Omega} e^{-i(\omega_{\mathbf{n}}(0)+\omega_{\mathbf{k}}(0))\tau} < c^2(\partial_\epsilon \dot{f}_{\mathbf{k}}(\tau, \mathbf{x}; 0)\nabla f_{\mathbf{n}}(\tau, \mathbf{x}; 0))_{|\partial\Omega}, d\mathbf{S} > \right|^2}{4\omega_{\mathbf{k}}(0)\omega_{\mathbf{n}}(0)(\omega_{\mathbf{k}}(0) + \omega_{\mathbf{n}}(0))^2} + \mathcal{O}(\epsilon^4). \tag{21}$$

For times greater than the stopping time, the average number of produced particles in the $\mathbf{n}$-mode is

$$\mathcal{N}_{\mathbf{n}}^{m}(t \geq T) \equiv \sum_{\mathbf{k}} \frac{\epsilon^2 \left| \int_0^T d\tau \int_{\partial\Omega} e^{-i(\omega_{\mathbf{n}}(0)+\omega_{\mathbf{k}}(0))\tau} < c^2(\partial_\epsilon \dot{f}_{\mathbf{k}}(\tau, \mathbf{x}; 0)\nabla f_{\mathbf{n}}(\tau, \mathbf{x}; 0))_{|\partial\Omega}, d\mathbf{S} > \right|^2}{4\omega_{\mathbf{k}}(0)\omega_{\mathbf{n}}(0)(\omega_{\mathbf{k}}(0) + \omega_{\mathbf{n}}(0))^2} + \mathcal{O}(\epsilon^4). \tag{22}$$

3. EXAMPLES (1+1 DIMENSIONAL CASE)

Consider the domain $\Omega_t = [0, L + \epsilon g(t)]$. In this situation the formula (21) behaves

$$\mathcal{N}_n^m(t) = \frac{\epsilon^2}{L^2}\left(\frac{c\pi}{L}\right)^4 \sum_{k=1}^{\infty} \frac{k^2 n^2 \left| \int_0^t \dot{g}(\tau) e^{i(\omega_n(0)+\omega_k(0))\tau} d\tau \right|^2}{\omega_n(0)\omega_k(0)(\omega_n(0) + \omega_k(0))^2} + \mathcal{O}(\epsilon^4), \tag{23}$$

where the frequencies are $\omega_n(0) = \frac{1}{\hbar}\sqrt{\frac{c^2\pi^2\hbar^2 n^2}{L^2} + m^2c^4}$.

Remark 3.1. From this formula, assuming that $g \in \mathcal{C}^1(\mathbb{R})$, if we integrate by parts we can easily show that the average number of produced "quasi-particles" at time $t, \mathcal{N}^m(t) \equiv \sum_{n=1}^{\infty} \mathcal{N}_n^m(t)$, is infinite. For times greater than the stopping time T, if $g \in \mathcal{C}^1(\mathbb{R} \setminus \{t_0\})$ the number of produced particles is also infinite (see for details ref. [1]).

For massless "quasi-particles" we have the formula

$$\mathcal{N}_n^0(t) = \frac{\epsilon^2}{L^2} \sum_{k=1}^{\infty} \frac{kn}{(n+k)^2} \left| \int_0^t \dot{g}(\tau) e^{i\frac{c\pi}{L}(n+k)\tau} d\tau \right|^2 + \mathcal{O}(\epsilon^4). \tag{24}$$

And the average number of produced "quasi-particles" at time t is

$$\mathcal{N}^0(t) \equiv \sum_{n=1}^{\infty} \mathcal{N}_n^0(t) = \frac{\epsilon^2}{6L^2} \sum_{j=1}^{\infty} \left(j - \frac{1}{j} \right) \left| \int_0^t \dot{g}(\tau) e^{i \frac{c\pi}{L} \tau j} d\tau \right|^2 + \mathcal{O}(\epsilon^4). \quad (25)$$

From this last formula we can see that the concept of particle, when the boundary moves, is ill-defined. Effectively, using the frequency cut-off $e^{-\frac{c\pi}{L} j\gamma}$ we get

$$\begin{aligned}
\mathcal{N}^0(t,\gamma) &\equiv \frac{\epsilon^2}{6L^2} \sum_{j=1}^{\infty} \left(j - \frac{1}{j} \right) \left| \int_0^t \dot{g}(\tau) e^{i \frac{c\pi}{L} \tau j} d\tau \right|^2 e^{-\frac{c\pi}{L} j\gamma} \\
&= -\frac{\epsilon^2}{6c^2\pi^2} \dot{g}^2(t) \ln\left(\frac{c\pi}{L}\gamma\right) \\
&\quad + \frac{\epsilon^2}{6} \int_0^t d\tau \int_0^\tau ds \ln\left(2 - 2\cos\left(\frac{c\pi}{L}(\tau - s)\right)\right) \\
&\quad \times \left[\frac{\dddot{g}(s)\dot{g}(\tau)}{c^2\pi^2} + \frac{\dot{g}(s)\dot{g}(\tau)}{L^2} \right] + \mathcal{O}\left(\frac{c\pi}{L}\gamma\right).
\end{aligned}$$

Then we can define the renormalised number of produced "quasi-particles" as follows:

$$\mathcal{N}_R^0(t) \equiv \frac{\epsilon^2}{6} \int_0^t d\tau \int_0^\tau ds \ln\left(2 - 2\cos\left(\frac{c\pi}{L}(\tau - s)\right)\right) \left[\frac{\dddot{g}(s)\dot{g}(\tau)}{c^2\pi^2} + \frac{\dot{g}(s)\dot{g}(\tau)}{L^2} \right].$$

From this formula we can deduce that, when the boundary moves, the renormalised number of produced "quasi-particles" does not have a defined sign (For example if we take $g(t) \sim L_0(w_0 t)^N$ for $t \in [0,\delta]$ with $0 < \delta \ll 1$ and $N \geq 3$, we have $\mathcal{N}_R^0(t) < 0$). Then we can conclude that, when the boundary moves, the "quasi-particles" are not veritable particles.

On the other hand, when the boundary returns at rest, if $g \in \mathcal{C}^2(\mathbb{R}) \cap \mathcal{C}^3[0,T]$, we find

$$\mathcal{N}^0(t \geq T) = \frac{\epsilon^2 L^2}{6c^4\pi^4} \sum_{j=1}^{\infty} \left(\frac{1}{j^3} - \frac{1}{j^5} \right) \left| \int_0^T \dddot{g}(\tau) e^{i \frac{c\pi}{L} \tau j} d\tau \right|^2 + \mathcal{O}(\epsilon^4).$$

That is, the number of produced particles is positive and finite.

3.1. Oscillating Boundaries: Creation of Massless Pairs

Example 3.1. In this first example we study the creation of massless particles when g has the form $g(t) = L\sin(2\omega_r(0)t)$, with $r \in \mathbb{N}$. We take the stopping time $T_N = \frac{2\pi}{\omega_1(0)} N$ with $N \in \mathbb{N}$, and we assume that $\epsilon\omega_1(0)T_N \ll 1$.

Remark 3.2. In this case the stopping time is a period of the function

$$\dot{g}(\tau)e^{i\frac{c\pi}{L}(n+k)\tau},$$

that appears in formula (24).

Then, from the formula (24) and using that the Hamiltonian is a periodic operator, we obtain the same result as Ji *et al.* (1996)

$$\mathcal{N}_n^0(t \geq T_N) = \begin{cases} \frac{1}{4}\,(\epsilon\omega_1(0)T_N)^2\,(2r-n)n + \mathcal{O}((\epsilon\omega_1(0)T_N)^4) & \text{for} \quad n < 2r \\ \mathcal{O}((\epsilon\omega_1(0)T_N)^4) & \text{for} \quad n \geq 2r. \end{cases} \tag{26}$$

And therefore, the number of produced particles is

$$\mathcal{N}^0(t \geq T_N) = \frac{1}{24}\,(\epsilon\omega_1(0)T_N)^2\,(2r-1)2r(2r+1) + \mathcal{O}((\epsilon\omega_1(0)T_N)^4). \tag{27}$$

Example 3.2. In this second example we consider the function $g(t) = L\sin(2\omega_1(0)t)$, and we suppose that the stopping time T_N is given by $T_N = \frac{\pi}{\omega_1(0)}(2N+1)$ with $N \in \mathbb{N}$ and $N \gg 1$. We also assume that $\epsilon\omega_1(0)T_N \ll 1$. From (24) we easily obtain

$$\mathcal{N}_1^0(t \geq T_N) \approx \frac{1}{4}\,(\epsilon\omega_1(0)T_N)^2\,; \qquad \frac{\mathcal{N}_n^0(t \geq T_N)}{\mathcal{N}_1^0(t \geq T_N)} \approx 0 \quad \text{for } n > 1. \tag{28}$$

On the other hand, it is not difficult to show that $\mathcal{N}^0(t \geq T_N) = \infty$. Consequently, we **cannot** make the approximation

$$\mathcal{N}_n^0(t \geq T_N) \approx \frac{1}{4}\,(\epsilon\omega_1(0)T_N)^2\,\delta_{1,n}, \tag{29}$$

because according to this approximation we have $\mathcal{N}^0\,(t \geq T_N) = \mathcal{N}_1^0\,(t \geq T_N)$, in contradiction with $\mathcal{N}^0\,(t \geq T_N) = \infty$.

Example 3.3. In this last example we would like to answer the question: When the approximation (29) is right? One answer would be when g satisfies the following assumptions:

$$\begin{aligned} &(1) - g \in \mathcal{C}^2(\mathbb{R}) \cap \mathcal{C}^3[0,T] \\ &(2) - g(t) = L\sin(2\omega_1(0)t) \quad \forall t \in \left[\frac{2\pi}{\omega_1(0)}, \frac{2\pi}{\omega_1(0)}N^*\right], \\ &\qquad \text{where } N^* \in \mathbb{N} \text{ satisfies } \frac{2\pi}{\omega_1(0)}N^* < T \leq \frac{2\pi}{\omega_1(0)}(N^*+1), \\ &(3) - T \gg 1 \quad \text{and} \quad \epsilon\omega_1(0)T \ll 1. \end{aligned} \tag{30}$$

With this hypothesis, for $n > 1$ the average number of produced pairs in the n-mode is:

$$\mathcal{N}_n^0\,(t \geq T) = \frac{\epsilon^2 L^2}{c^4\pi^4}\sum_{k=1}^{\infty}\frac{kn}{(k+n)^6}\left|\int_I \dddot{g}\,(\tau)e^{i\frac{c\pi}{L}(n+k)\tau}d\tau\right|^2 + \mathcal{O}((\epsilon\omega_1(0)T)^4), \quad (31)$$

where $I = [0, \frac{2\pi}{\omega_1(0)}] \cup [\frac{2\pi}{\omega_1(0)}N^*, T]$. And then

$$\sum_{n=2}^{\infty}\mathcal{N}_n^0(t \geq T) \leq \frac{6\epsilon^2 L^4}{c^6\pi^4}||\,\dddot{g}\,||_\infty^2(\zeta_R(3) - \zeta_R(5)), \quad (32)$$

where ζ_R is the Riemann zeta function.

On the other hand, for $n = 1$ we have

$$\begin{aligned}\mathcal{N}_1^0(t \geq T) &= \frac{1}{4}\,(\epsilon\omega_1(0))^2\left(\frac{2L}{c}(N^* - 1)\right)^2\\ &\quad + \frac{\epsilon^2 L^2}{c^4\pi^4}\sum_{k=1}^{\infty}\frac{k}{(k+1)^6}\left|\int_I \dddot{g}\,(\tau)e^{i\frac{c\pi}{L}(1+k)\tau}d\tau\right|^2\\ &\quad - \frac{\epsilon^2 L}{4c^2\pi}(N^* - 1)\int_I \dddot{g}\,(\tau)\cos(2\omega_1(0)\tau)d\tau + \mathcal{O}((\epsilon\omega_1(0)T)^4). \quad (33)\end{aligned}$$

Consequently, since $T \gg 1$ and $\frac{2L}{c}(N^* - 1) \approx T$, we have

$$\mathcal{N}_1^0(t \geq T) \approx \frac{\epsilon^2}{4}\left(\frac{c\pi}{L}T\right)^2; \quad \frac{\sum_{n=2}^{\infty}\mathcal{N}_n^0(t \geq T)}{\mathcal{N}_1^0(t \geq T)} \approx 0. \quad (34)$$

And thus, when g satisfies (30), the approximation (29) holds.

Remark 3.3. From this last example, we conclude that the formula (4.6) of the Ji *et al.* (1996) is valid for $t \geq T$, only when g satisfies a similar version of the assumption (30).

3.2. Rotating Wave Approximation

Here we will calculate, for large times, the average number of produced particles in the massless case. When the Hamiltonian in the interaction picture is a periodic operator, we can use the so-called "Rotating wave approximation," based in the following approximation:

Firstly using the time-ordering operator $\hat{\mathcal{T}}$, we can write the evolution operator in the form

$$\mathcal{T}^t = \mathcal{T}_0^t\hat{\mathcal{T}}\exp\left(-\frac{i}{\hbar}\int_0^t \hat{W}_I(\tau;\epsilon)d\tau\right), \quad (35)$$

where $\hat{W}_I(\tau;\epsilon) \equiv \hat{H}_I(\tau;\epsilon) - \hat{H}_I(\tau;0)$ taken in the Interaction picture.

Let $\widetilde{\tau}$ be the period of $\hat{W}_I(\tau;\epsilon)$, then the RWA is based in the approximation (see Thimmel *et al.*, 1999)

$$\mathcal{T}^t \approx \mathcal{T}_0^t e^{-\frac{i}{\hbar}\hat{W}_{\text{eff}}(\epsilon)t}, \tag{36}$$

where $\hat{W}_{\text{eff}}(\epsilon) \equiv \frac{1}{\widetilde{\tau}}\int_0^{\widetilde{\tau}} \hat{W}_I(\tau;\epsilon)\,d\tau$

Remark 3.4. When $t = \widetilde{\tau}N$ with $N \in \mathbb{N}$, we have exactly

$$\mathcal{T}^t = \mathcal{T}_0^t e^{-\frac{i}{\hbar}\hat{W}_{\text{eff}}(\epsilon)t}.$$

Here we make another approximation. The operator $e^{-\frac{i}{\hbar}\hat{W}_{\text{eff}}(\epsilon)\widetilde{\tau}N}$ can be expanded in power series of the dimensionless parameters ϵ and N in the form

$$e^{-\frac{i}{\hbar}\hat{W}_{\text{eff}}(\epsilon)\widetilde{\tau}N} = \sum_{\substack{r,s\\ r\geq s}} \hat{A}_{r,s}\epsilon^r N^s. \tag{37}$$

Thus, in the case that $\epsilon N \sim \mathcal{O}(1)$ or $\epsilon N \ll 1$, we can only retain the terms $(\epsilon N)^n$, and it is not difficult to show that (see Schützhold *et al.*, 2002; Schaller *et al.*, 2002)

$$e^{-\frac{i}{\hbar}\hat{W}_{\text{eff}}(\epsilon)\widetilde{\tau}N} \approx e^{-\frac{i}{\hbar}\epsilon\hat{H}_{\text{eff}}\widetilde{\tau}N}, \tag{38}$$

where the effective Hamiltonian is defined by $\hat{H}_{\text{eff}} \equiv \frac{1}{\widetilde{\tau}}\int_0^{\widetilde{\tau}} \partial_\epsilon \hat{W}_I(\tau;0)d\tau$. Finally, assuming that the stopping time is $\widetilde{\tau}N$, our RWA approximation that suits when $\epsilon N \ll 1$ or $\epsilon N \sim \mathcal{O}(1)$, is

$$\mathcal{T}^t \approx \mathcal{T}_0^t e^{-\frac{i}{\hbar}\epsilon\hat{H}_{\text{eff}}\widetilde{\tau}N} \qquad \forall t \geq N\widetilde{\tau}. \tag{39}$$

Example 3.4. If we take $g(t) = L\sin(2\omega_1(0)t)$ for $t \in (0, \frac{2\pi}{\omega_1(0)}N)$, we have

$$\hat{H}_{\text{eff}} = \frac{-i\hbar}{4}\omega_1(0)\left[(\hat{a}_1^\dagger)^2 - (\hat{a}_1)^2 + 2\sum_{n=1}^{\infty}\sqrt{n(n+2)}(\hat{a}_{n+2}^\dagger\hat{a}_n - \hat{a}_n^\dagger\hat{a}_{n+2})\right] \tag{40}$$

Thus, in the rotating wave approximation, the average number of massless particles in the l-mode is

$$\mathcal{N}_l^0\,(t \geq T_N) \approx \langle 0|\exp\left(\frac{i}{\hbar}\epsilon\hat{H}_{\text{eff}}T_N\right)\hat{a}_l^\dagger\hat{a}_l\exp\left(-\frac{i}{\hbar}\epsilon\hat{H}_{\text{eff}}T_N\right)|0\rangle, \tag{41}$$

where $T_N \equiv \frac{2\pi}{\omega_1(0)}N$.

Now, if we define

$$\hat{c}_l(s) \equiv \exp\left(\frac{i}{\hbar}\epsilon\hat{H}_{\text{eff}}s\right)\hat{a}_l\exp\left(-\frac{i}{\hbar}\epsilon\hat{H}_{\text{eff}}s\right),$$

we can write

$$\mathcal{N}_l^0\,(t \geq T_N) \approx \langle 0|\hat{c}_l^\dagger(T_N)\hat{c}_l(T_N)|0\rangle. \tag{42}$$

The operators $\hat{c}_l(s)$ satisfy the Heisenberg equation $\dot{\hat{c}}_l(s) = \epsilon \frac{i}{\hbar}[\hat{H}_{\text{eff}}, \hat{c}_l(s)]$. In our case, these equations are

$$\begin{cases} \dot{\hat{c}}_1 = \frac{\epsilon\omega_1(0)}{2}(-\hat{c}_1^\dagger + \sqrt{3}\hat{c}_3) \\ \dot{\hat{c}}_l = \frac{\epsilon\omega_1(0)}{2}(-\sqrt{l(l-2)}\hat{c}_{l-2} + \sqrt{l(l+2)}\hat{c}_{l+2}) \quad \text{for} \quad l \geq 2. \end{cases} \tag{43}$$

Now, following the Dodonov and Klimov (1996), we write

$$\hat{c}_r(s) = \sum_{n=1}^{\infty} \hat{a}_n \sqrt{\frac{r}{n}} \xi_r^{(n)}(s) + \hat{a}_n^\dagger \sqrt{\frac{r}{n}} \left(\eta_r^{(n)}(s)\right)^*, \tag{44}$$

and we introduce the dimensionless parameter $\mu = \frac{\epsilon\omega_1(0)}{2}s$, then inserting (44) into (43) we obtain the equation (3.2)–(3.5) of Dodonov and Klimov (1996).

And therefore, we have (see formula (6.5) of Dodonov and Klimov, 1996)

$$\mathcal{N}_1^0(t \geq T_N) \approx \frac{2}{\pi^2} E(\sqrt{1 - e^{-8\mu_N}}) K(\sqrt{1 - e^{-8\mu_N}}) - \frac{1}{2}, \tag{45}$$

where $\mu_N \equiv \frac{\epsilon\omega_1(0)}{2} T_N$ and, E and K are the elliptic integrals

$$E(x) = \int_0^{\frac{\pi}{2}} d\alpha \sqrt{1 - x^2 \sin^2\alpha}; \quad K(x) = \int_0^{\frac{\pi}{2}} \frac{d\alpha}{\sqrt{1 - x^2 \sin^2\alpha}}.$$

Then when $\mu_N \ll 1$ we obviously obtain (26) with $r = 1$. On the other hand, when $\mu_N \gg 1$ the formula (45) provides (see (6.7) of Dodonov and Klimov, 1996)

$$\mathcal{N}_1^0\ (t \geq T_N) \approx \frac{8}{\pi^2}\mu_N + \frac{2}{\pi^2}\ln 4 - \frac{1}{2}. \tag{46}$$

However we cannot ensure the correctness of this formula because our approximation is only valid when $\mu_N \ll 1$ or $\mu_N \sim \mathcal{O}(1)$. Moreover, according to (46) we have $\mathcal{N}_1^0(t \geq T_N) \gg 1$, this shows that particles created by oscillating boundaries could easily detected, but we do not know any experiment that measures the dynamical Casimir effect.

Remark 3.5. If we take the stopping time $T_N = \frac{\pi}{\omega_1(0)}(2N+1)$ (in this case $T_N \neq \tilde{\tau}N$), our RWA is a bad approximation. Effectively, we have

$$\exp\left(\epsilon \frac{i}{\hbar} \hat{H}_{\text{eff}} T_N\right) = 1 - \epsilon \frac{i}{\hbar} \hat{H}_{\text{eff}} T_N + \cdots,$$

and then

$$\begin{aligned} \mathcal{N}^0\ (t \geq T_N) &\approx \frac{\epsilon^2 T_N^2}{\hbar^2} \sum_{n=1}^{\infty} \langle 0|\hat{H}_{\text{eff}} \hat{a}_n^\dagger(T_N) \hat{a}_n(T_N) \hat{H}_{\text{eff}}|0\rangle + \cdots \\ &= \frac{1}{4}(\epsilon\omega_1(0)T_N)^2 + \mathcal{O}((\epsilon\omega_1(0)T_N)^4), \end{aligned}$$

but in the same way as the example 3.2, we can deduce that, the term of order ϵ^2 of the number of produced particles at times greater than T_N is divergent.

To obtain the result (45) for times $T \neq \tilde{\tau} N$ with $\epsilon\omega_1(0)T \sim \mathcal{O}(1)$ we must make a similar version of the assumption (30) (see for more details the Section 4.2).

3.3. Energy Calculation in the RWA: 1+1-Dimensional Case

Here we calculate the energy of the system studied in the example 3.4. The energy of this system has been calculated in the Dodonov and Klimov (1996). Here using the RWA we will obtain, in an easier way, the same result.

The radiated energy is

$$\langle \hat{E}^0(T_N)\rangle \equiv \sum_{k=1}^{\infty} \hbar\omega_k(0)\mathcal{N}_k^0(s \geq T_N) = \hbar\omega_1(0) \sum_{r,n=1}^{\infty} \frac{r^2}{n} \left|\eta_r^{(n)}(T_N)\right|^2,$$

with $T_N = \frac{2\pi}{\omega_1(0)} N$.

For $s \in [0, T_N]$ we consider the quantity

$$\langle \hat{E}^0(s)\rangle \equiv \hbar\omega_1(0) \sum_{r,n=1}^{\infty} \frac{r^2}{n} \left|\eta_r^{(n)}(s)\right|^2, \tag{47}$$

and following the Dodonov and Klimov (1996) we define the function $S^{(n)}(s) = \sum_{r=1}^{\infty} r^2 |\eta_r^{(n)}(s)|^2$, that satisfies the differential equation

$$\ddot{S}^{(n)} = \frac{\epsilon\omega_1(0)}{2} \left(\xi_1^{(n)}\left(\dot{\eta}_1^{(n)}\right)^* + \left(\xi_1^{(n)}\right)^* \dot{\eta}_1^{(n)} - \dot{\xi}_1^{(n)}\left(\eta_1^{(n)}\right)^* - \left(\dot{\xi}_1^{(n)}\right)^* \eta_1^{(n)}\right)$$
$$+ \left(\frac{\epsilon\omega_1(0)}{2}\right)^2 \left[4\left(\left|\xi_1^{(n)}\right|^2 - \left|\eta_1^{(n)}\right|^2\right) + 16 S^{(n)}\right]. \tag{48}$$

The operators $\hat{c}(s)$ and $\hat{c}^\dagger(s)$ defined in the example 3.4 satisfies the commutation rules

$$[\hat{c}_j(s), \hat{c}_k(s)] = 0; \quad [\hat{c}_j^\dagger(s), \hat{c}_k(s)] = -\delta_{j,k},$$

then, from the first equation of (43), and from the equations (3.2)–(3.4) of the Dodonov and Klimov (1996), if we use these commutation rules, we obtain the relation

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\dot{\xi}_1^{(n)}\left(\eta_1^{(n)}\right)^* - \xi_1^{(n)}\left(\dot{\eta}_1^{(n)}\right)^*\right) = \frac{\epsilon\omega_1(0)}{2}. \tag{49}$$

From the equation $[\hat{c}_1^\dagger(s), \hat{c}_1(s)] = -1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\left|\xi_1^{(n)}\right|^2 - \left|\eta_1^{(n)}\right|^2 \right) = 1. \tag{50}$$

Thus, for $s \in [0, T_N]$, the function $\langle \hat{E}^0(s) \rangle$ satisfies the equation

$$\frac{d^2}{ds^2} \langle \hat{E}^0(s) \rangle = \left(\frac{\epsilon \omega_1(0)}{2} \right)^2 \left(16 \langle \hat{E}^0(s) \rangle + 2\hbar\omega_1(0) \right). \tag{51}$$

And therefore, since $\langle \hat{E}^0(0) \rangle = \frac{d}{ds} \langle \hat{E}^0(0) \rangle = 0$, when $\epsilon\omega_1(0)T_N \sim \mathcal{O}(1)$ or $\epsilon\omega_1(0)T_N \ll 1$, in the RWA the total radiated energy is

$$\langle \hat{E}^0(T_N) \rangle = \frac{1}{4} \hbar\omega_1(0) \sinh^2(\epsilon\omega_1(0)T_N). \tag{52}$$

4. EXAMPLES (3 + 1 DIMENSIONAL CASE)

In this Section we consider a rectangular cavity with a moving wall, that is, a volume of this form

$$\Omega_t = [0, L_1 + \epsilon g(t)] \times [0, L_2] \times [0, L_3].$$

In this case the formula (21) behaves

$$\mathcal{N}_{\mathbf{n}}^m(t) = \frac{\epsilon^2}{{L_1}^2} \left(\frac{c\pi}{L_1} \right)^4 \sum_{\mathbf{k} \in \mathbb{N}^3}^{\infty} \frac{k_1^2 n_1^2 \left| \int_0^t \dot{g}(\tau) e^{i(\omega_{\mathbf{n}}(0) + \omega_{\mathbf{k}}(0))\tau} d\tau \right|^2}{\omega_{\mathbf{n}}(0)\omega_{\mathbf{k}}(0)(\omega_{\mathbf{n}}(0) + \omega_{\mathbf{k}}(0))^2} \delta_{k_2,n_2} \delta_{k_3,n_3} + \mathcal{O}(\epsilon^4), \tag{53}$$

where the frequencies are

$$\omega_{\mathbf{n}}(0) = \frac{1}{\hbar} \sqrt{c^2\pi^2\hbar^2 \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} + \frac{n_3^2}{L_3^2} \right) + m^2c^4}.$$

Remark 4.1. From this formula, assuming that $g \in \mathcal{C}^2(\mathbb{R})$, if we integrate by parts we can easily deduce that, the average number of produced "quasi-particles" at time t, $\mathcal{N}^m(t) \equiv \sum_{n=1}^{\infty} \mathcal{N}_n^m(t)$, is infinite. For times greater than the stopping time T, if $g \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R} \setminus \{t_0\})$, the number of produced particles is also infinite.

4.1. Oscillating Boundaries

If we take the function $g(t) = L_1 \sin(\omega_{\mathbf{n}}(0)t)$, an easy calculation provides that

$$\mathcal{N}_{\mathbf{n}}^{m}(t \geq T) = \infty \quad \forall \mathbf{n} \in \mathbb{N}^3. \tag{54}$$

Consequently, to obtain a similar result to the obtained in formula (29) we need some assumptions. We will suppose that

(1) $-\ g \in \mathcal{C}^3(\mathbb{R}) \cap \mathcal{C}^4[0, T]$

(2) $-\ g(t) = L \sin(2\omega_{(1,1,1)}(0)t) \quad \forall t \in \left[\frac{\pi}{\omega_{(1,1,1)}(0)}, \frac{\pi}{\omega_{(1,1,1)}(0)} N^*\right],$

where $N^* \in \mathbb{N}$ satisfies

$$\frac{\pi}{\omega_{(1,1,1)}(0)} N^* < T \leq \frac{\pi}{\omega_{(1,1,1)}(0)} (N^* + 1)$$

(3) $-\ T \gg 1 \quad \text{and} \quad \epsilon\omega_{(1,1,1)}(0)T \ll 1.$ (55)

Now assuming (55) we can check that

$$\sum_{\substack{\mathbf{n} \in \mathbb{N}^3 \\ \mathbf{n} \neq (1,1,1)}} \mathcal{N}_{\mathbf{n}}^{m}(t \geq T) \leq \frac{K\epsilon^2 L^6}{c^8} ||\ddddot{g}||_\infty + \mathcal{O}(\epsilon^4), \tag{56}$$

where K is a constant independent on N^*. We also obtain

$$\mathcal{N}_{(1,1,1)}^{m}(t \geq T) = \frac{1}{36} \big(\epsilon\omega_{(1,1,1)}(0)\big)^2 \left(\frac{N^* - 1}{\omega_{(1,1,1)}(0)} \pi\right)^2 + R + \mathcal{O}(\epsilon^4), \tag{57}$$

with

$$|R| \leq \frac{C\epsilon^2 (N^* - 1) L^3}{c^8} ||\ddddot{g}||_\infty,$$

where C is a constant independent on N^*. Thus, since $T \approx \frac{\pi}{\omega_{(1,1,1)}(0)}(N^* - 1)$, we have

$$\frac{\sum_{\substack{\mathbf{n} \in \mathbb{N}^3 \\ \mathbf{n} \neq (1,1,1)}} \mathcal{N}_{\mathbf{n}}^{m}(t \geq T)}{\mathcal{N}_{(1,1,1)}^{m}(t \geq T)} \approx 0;$$

$$\mathcal{N}_{(1,1,1)}^{m}(t \geq T) \approx \frac{1}{36} (\epsilon\omega_{(1,1,1)}(0)T)^2, \tag{58}$$

and therefore, we conclude that

$$\mathcal{N}_{\mathbf{n}}^{m}(t \geq T) \approx \frac{1}{36} \big(\epsilon\omega_{(1,1,1)}(0)T\big)^2 \delta_{\mathbf{n},(1,1,1)}. \tag{59}$$

4.2. Rotating Wave Approximation

In the 3 + 1-dimensional case, to obtain the number of created particles for large times, we make the assumptions 1 and 2 of (55) with $\omega = 2\omega_{(1,1,1)}$

If we suppose that $T \gg 1$ and, $\epsilon\omega T \ll 1$ or $\epsilon\omega T \sim \mathcal{O}(1)$. For $t \geq T$, we can make the approximation

$$\mathcal{T}^t \approx \mathcal{T}_0^t e^{-\frac{i}{\hbar}\epsilon \hat{H}_{\text{eff}} T}. \tag{60}$$

where we have defined $\hat{H}_{\text{eff}} \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T \partial_\epsilon \hat{W}_I(\tau;0)\, d\tau$ (see for details Schützhold *et al.* (2002) and Schaller *et al.* (2002)).

Example 4.1. In this first example we take $\omega = 2\omega_{\mathbf{r}}(0)$. We assume that L_1 is a transcendent number and, L_2 and L_3 are rational numbers. In this case, there not exist any mode $\mathbf{n}$ and $\mathbf{k}$ with $n_2 = k_2$ and $n_3 = k_3$ such that $2\omega_{\mathbf{r}}(0) = \omega_{\mathbf{n}}(0) \pm \omega_{\mathbf{k}}(0)$.

Then, we have

$$\mathcal{T}^t \approx \mathcal{T}_0^t \exp\left(\frac{1}{4}\epsilon\frac{\pi^2 c^2 r_1^2/L_1^2}{\omega_{\mathbf{r}}(0)} T[(\hat{a}_{\mathbf{r}})^2 - (\hat{a}_{\mathbf{r}}^\dagger)^2]\right), \tag{61}$$

Now taking into account the formula (see Lo and Sollie, 1993)

$$\hat{S}_{\mathbf{n}}^\dagger(\beta)\hat{a}_{\mathbf{n}}\hat{S}_{\mathbf{n}}(\beta) = \cosh(|\beta|)\hat{a}_{\mathbf{n}} + \frac{\beta}{|\beta|}\sinh(|\beta|)\hat{a}_{\mathbf{n}}^\dagger, \tag{62}$$

where

$$\hat{S}_{\mathbf{n}}(\beta) = \exp\left(\frac{1}{2}[\beta(\hat{a}_{\mathbf{n}}^\dagger)^2 - \beta^*(\hat{a}_{\mathbf{n}})^2]\right),$$

we obtain, that the average number of produced particles in the $\mathbf{s}$-mode, is

$$\mathcal{N}_{\mathbf{s}}^m(t \geq T) \approx \sinh^2\left(\frac{1}{2}\epsilon\frac{\pi^2 c^2 s_1^2/L_1^2}{\omega_{\mathbf{s}}(0)} T\right)\delta_{\mathbf{s},\mathbf{r}}. \tag{63}$$

Example 4.2. Another interesting example is when $\omega = w_{\mathbf{r}}(0) + w_{\mathbf{s}}(0)$, where $\mathbf{r}$ and $\mathbf{s}$ are two non-resonant modes that satisfies $r_2 = s_2$ and $r_3 = s_3$. We also assume that there is not another mode coupled with $\mathbf{r}$ and $\mathbf{s}$, that is, there not exist any mode $\mathbf{n}$ that satisfies $\omega_{\mathbf{n}}(0) = 2w_{\mathbf{r}}(0) + w_{\mathbf{s}}(0)$ or $\omega_{\mathbf{n}}(0) = w_{\mathbf{r}}(0) + 2w_{\mathbf{s}}(0)$.

In this situation we have

$$\mathcal{T}^t \approx \mathcal{T}_0^t \exp\left(-\frac{\epsilon}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{r_1 s_1}{\sqrt{w_{\mathbf{r}}(0)w_{\mathbf{s}}(0)}}(-1)^{r_1+s_1} T[\hat{a}_{\mathbf{r}}^\dagger\hat{a}_{\mathbf{s}}^\dagger - \hat{a}_{\mathbf{r}}\hat{a}_{\mathbf{s}}]\right)\exp\left(\hat{O}\right), \tag{64}$$

where $\hat{O}$ is an operator that does not contain the creation and annihilation operators of the modes $\mathbf{r}$ and $\mathbf{s}$.

If we use the formulae (8) and (9) of (Lo and Sollie, 1993) the average number of produced particles in the **r**-mode and in the **s**-mode are

$$\mathcal{N}_{\mathbf{r}}^{m}(t \geq T) = \mathcal{N}_{\mathbf{s}}^{m}(t \geq T) \approx \sinh^2\left(\epsilon\left(\frac{\pi c}{L_1}\right)^2 \frac{r_1 s_1}{2\sqrt{w_{\mathbf{r}}(0) w_{\mathbf{s}}(0)}} T\right). \quad (65)$$

Example 4.3. In this example we consider the case $\omega = 2\omega_{\mathbf{r}}(0)$, and we assume that there exist one mode **s**, and only one, coupled with **r**, that is, a mode that satisfies $\omega_{\mathbf{s}}(0) = 3\omega_{\mathbf{r}}(0)$ with $r_2 = s_2$ and $r_3 = s_3$. In this case, we have

$$\hat{H}_{\text{eff}} = \frac{i\hbar}{4} \frac{\pi^2 c^2 r_1^2 / L_1^2}{\omega_{\mathbf{r}}(0)} [(\hat{a}_{\mathbf{r}})^2 - (\hat{a}_{\mathbf{r}}^{\dagger})^2] - \frac{i\hbar}{2\sqrt{3}} \frac{\pi^2 c^2 / L_1^2}{\omega_{\mathbf{r}}(0)} s_1 r_1 (-1)^{s_1 + r_1} [\hat{a}_{\mathbf{s}}^{\dagger} \hat{a}_{\mathbf{r}} - \hat{a}_{\mathbf{r}}^{\dagger} \hat{a}_{\mathbf{s}}] + i\hbar \hat{O}, \quad (66)$$

where $\hat{O}$ is an operator that does not contain the creation and annihilation operators of the modes **r** and **s**.

In the same way of the example 3.4 we define the operators

$$\hat{c}_{\mathbf{l}}(s) \equiv \exp\left(\frac{i}{\hbar}\epsilon \hat{H}_{\text{eff}} s\right) \hat{a}_{\mathbf{l}} \exp\left(-\frac{i}{\hbar}\epsilon \hat{H}_{\text{eff}} s\right),$$

Now, following the Crocce *et al.* (2002) if we write

$$\hat{c}_{\mathbf{k}}(s) = \sum_{\mathbf{n}} \hat{a}_{\mathbf{k}} \sqrt{2\omega_{\mathbf{k}}} B_{\mathbf{k}}^{(\mathbf{n})}(s) + \hat{a}_{\mathbf{k}}^{\dagger} \sqrt{2\omega_{\mathbf{k}}} \left(A_{\mathbf{k}}^{(\mathbf{n})}(s)\right)^*,$$

and we introduce the new time $\tau = \epsilon s$, we obtain the equations (41) and (42) of the Crocce *et al.* (2002).

An example of this situation is the massless Klein-Gordon field in a cubic cavity. Two coupled modes are $\mathbf{r} = (1, 1, 1)$ and $\mathbf{s} = (5, 1, 1)$. Thus, we obtain the equations (54) and (55) of the Crocce *et al.* (2002) with $\tau_f = \epsilon \frac{cT}{L}$.

Example 4.4. In this last example we study the case $\omega = \omega_{\mathbf{s}}(0) - \omega_{\mathbf{r}}(0)$, with $\omega_{\mathbf{s}}(0) \neq 3\omega_{\mathbf{r}}(0)$ and $r_2 = s_2$ and $r_3 = s_3$. We also suppose that there not exist any mode **n** such that $\omega_{\mathbf{s}}(0) = \omega_{\mathbf{n}}(0) + 2\omega_{\mathbf{r}}(0)$. In this situation the effective Hamiltonian is

$$\hat{H}_{\text{eff}} = -\frac{i\hbar}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)} (-1)^{s_1 + r_1} [\hat{a}_{\mathbf{s}}^{\dagger} \hat{a}_{\mathbf{r}} - \hat{a}_{\mathbf{r}}^{\dagger} \hat{a}_{\mathbf{s}}] + i\hbar \hat{O}, \quad (67)$$

where $\hat{O}$ is an operator that does not contain the creation and annihilation operators of the modes **r** and **s**.

From the operators

$$\hat{c}_{\mathbf{l}}(s) = \exp\left(\frac{i}{\hbar}\epsilon \hat{H}_{\text{eff}} s\right) \hat{a}_{\mathbf{l}} \exp\left(-\frac{i}{\hbar}\epsilon \hat{H}_{\text{eff}} s\right),$$

we obtain the following system of equations

$$\begin{cases} \dot{\hat{c}}_{\mathbf{s}} = -\frac{1}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)}(-1)^{s_1+r_1}\hat{c}_{\mathbf{r}} \\ \dot{\hat{c}}_{\mathbf{r}} = \frac{1}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)}(-1)^{s_1+r_1}\hat{c}_{\mathbf{s}}. \end{cases} \tag{68}$$

The solution of this system is

$$\hat{c}_{\mathbf{s}}(s) = \hat{a}_{\mathbf{s}} \cos\left(\frac{1}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1(-1)^{s_1+r_1}}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)} s\right) - \hat{a}_{\mathbf{r}} \sin\left(\frac{1}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1(-1)^{s_1+r_1}}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)} s\right)$$

$$\hat{c}_{\mathbf{r}}(s) = \hat{a}_{\mathbf{r}} \cos\left(\frac{1}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1(-1)^{s_1+r_1}}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)} s\right) + \hat{a}_{\mathbf{s}} \sin\left(\frac{1}{2}\left(\frac{\pi c}{L_1}\right)^2 \frac{s_1 r_1(-1)^{s_1+r_1}}{\omega_{\mathbf{r}}(0)\omega_{\mathbf{s}}(0)} s\right).$$

Thus, we conclude that

$$\mathcal{N}_{\mathbf{r}}(t \geq T) = \mathcal{N}_{\mathbf{s}}(t \geq T) = 0. \tag{69}$$

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